Simplified Solution to the Eddy-Viscosity Wake Model

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Revision History

Issue	Date	Author	Nature And Location Of Change
01	10 Jul 2009	Mike Anderson	First Created
02	14 Jul 2011	Mike Anderson	Additional equations.

CONTENTS

1.0 INTRODUCTION

This report describes a numerically efficient solution of the Ainslie [1] eddy-viscosity model.

2.0 AINSLIE WAKE MODEL

As given by Ainslie [1] the wake behind a wind turbine requires the solution of the thin shear layer Navier-Stokes equation

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial r}=\frac{\varepsilon}{r}\left(\frac{\partial U}{\partial r}+r\frac{\partial^2 U}{\partial r^2}\right) \qquad [1]$$

and the equation of continuity

$$\frac{\partial U}{\partial x}=-\frac{1}{r}\left(r\frac{\partial V}{\partial r}+V\right) \qquad [2]$$

where

U	is the velocity in the downstream direction.
V	is the velocity in the radial direction.
x	is the distance downstream from the rotor.
r	is the radial distance from the rotor centre line.
ε	is the eddy-viscosity.

The eddy-viscosity is calculated as

$$\varepsilon = F\left(K_1 b\left(U_0-U_c\right)+K_m\right)$$

$$K_m=\kappa^2\frac{I_0}{100}$$

$F=1.0$ for $x \geq 5.5$ and

$$F=0.65+\left(\frac{x-4.5}{23.32}\right)^{1/3} \quad x<5.5$$

where K_1 is a dimensionless constant set equal to 0.015, κ is the Von Karman constant, I_0 is the ambient turbulence level expressed as a percentage, U_c is the centre line velocity, U_o is the free stream velocity and b is a measure of the wake width. Based on experimental data Ainslie concluded that the initial wake profile is Gaussian and of the form

$$1-\frac{U}{U_0}=D_M e^{(-3.56(r/b)^2)} \qquad [3]$$

Through the conservation of momentum it is possible to relate the centreline wake deficit, D_M to the rotor thrust coefficient, C_t and hence arrive at an expression for the initial wake width

$$C_T = \frac{T}{\frac{1}{2}\rho U_0^{\ 2} A} = \frac{2}{A}\int \frac{U}{U_0}(1-\frac{U}{U_0})dA = \frac{2}{A}\int_0^{\infty}\int_0^{2\pi} \frac{U}{U_0}(1-\frac{U}{U_0})rdrd\theta \qquad [4]$$

Substituting equation [3], integrating and solving for b we have

$$b = \sqrt{\frac{3.56C_T}{(8D_M(1-0.5D_M)}}$$

It should be noted that all distances have been normalised with respect to the rotor diameter. From experimental data Ainslie derived an empirical relationship which relates the initial wake deficit to the rotor thrust coefficient and the ambient turbulence intensity

$$D_{M_i} = C_t - 0.05 - (16C_t - 0.5)\frac{I_o}{1000}$$

It is now possible to solve for the velocity behind a wind turbine. The solution of these equations requires a numerical integration scheme such as the Crank-Nicolson method. The Crank–Nicolson method, which unconditionally stable is based on a central difference in space and the trapezoidal rule in time, giving second-order convergence in time. To ensure that the wake is allowed to expand, without constraint requires that the integration domain is an order of magnitude larger than the rotor diameter. As a consequence the solution of equation can be very time consuming and for this reason an alternative approach has been developed.

3.0 SIMPLE MODEL

Solution of the above equations revealed that wake profile is self similar at all distances downwind. In other words the initial Gaussian shape is preserved and only its width and depth change. As the wake width is related to the wake deficit through conservation of momentum we are left with only the wake centreline velocity deficit to solve for.

From equation [3] we have

$$U = U_0\left(1 - D_M e^{(-3.56(r/b)^2)}\right)$$

Substituting this and equation [2] into equation [1] and noting that we are only required to solve for the centreline velocity we obtain

$$\frac{dU_c}{dx} = \frac{16\varepsilon\left(U_c^3 - U_c^2 - U_c + 1\right)}{U_c C_t}$$

where

$$U_c = U_0\left(1 - D_M\right)$$

As this equation is a first order differential equation of the form

$$\frac{dy}{dz} = f(y, x)$$

it may be solved efficiently using a simple numerical integration scheme such as Runge Kutta.

4.0 VALIDATION

A comparison of three methods for determining the wake profile behind a wind turbine are presented in Figures 1 to 3 for a range of ambient turbulence intensities and thrust coefficients. The three methods are:

- Solution using Crank-Nicolson integration scheme.
- Solution currently used in WFYIELD.
- Simple solution.

From the figures it can be observed that all three methods give very similar results. It should be noted that the Crank-Nicolson scheme produces a slightly greater velocity deficit than the other two and this is thought to be due to the finite domain volume constraining the wake growth. This was confirmed by integrating the wake profile and noting that the computed rotor thrust coefficient increased as you moved downstream, see Figure 4. The integration method in WFYIELD does not suffer from the same problem as the integration scheme “forces” momentum to be conserved.

5.0 CONCLUSIONS

The simplified solution of the Ainslie wake eddy-viscosity equation provides an elegant method for rapidly computing the wake profile behind a wind turbine. The model allows the parameters to be changed at run time, as opposed to using a look table as implemented in WFYIELD thereby allowing atmospheric stability and turbulence variations to be easily accommodated.

6.0 REFERENCES

1. Ainslie, J. F. “Calculating the Flowfield in the Wake of Wind Turbines”, Journal of Wind Engineering and Industrial Aerodynamics, Volume 27, pp. 213-224, 1988.

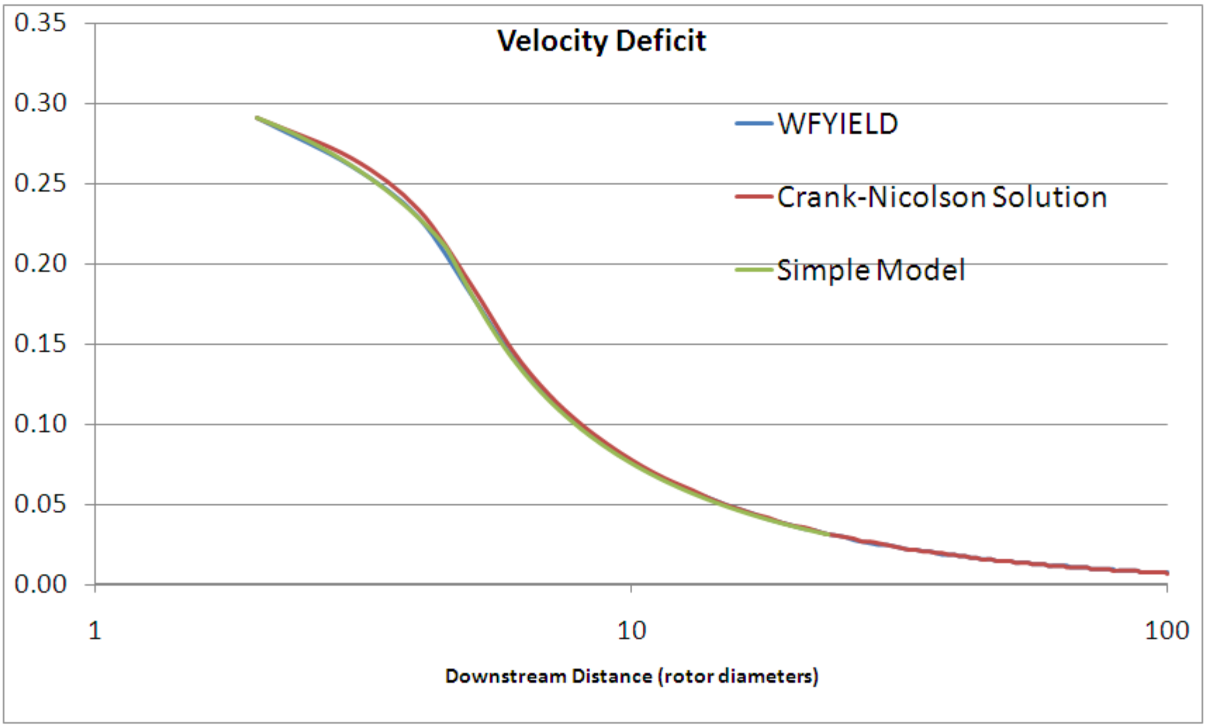

Figure 1 – Variation of Centreline Wake Deficit with Downstream Distance (Ct = 0.4, I = 10%).

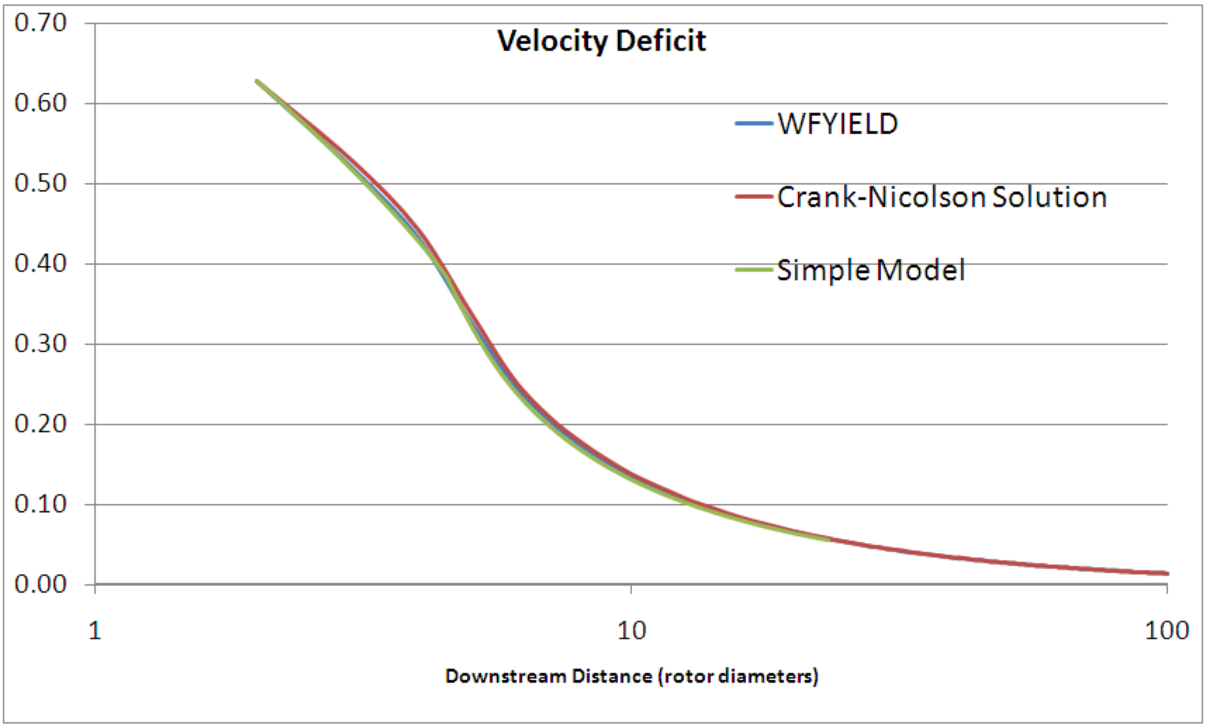

Figure 2 – Variation of Centreline Wake Deficit with Downstream Distance (Ct = 0.8, I = 10%).

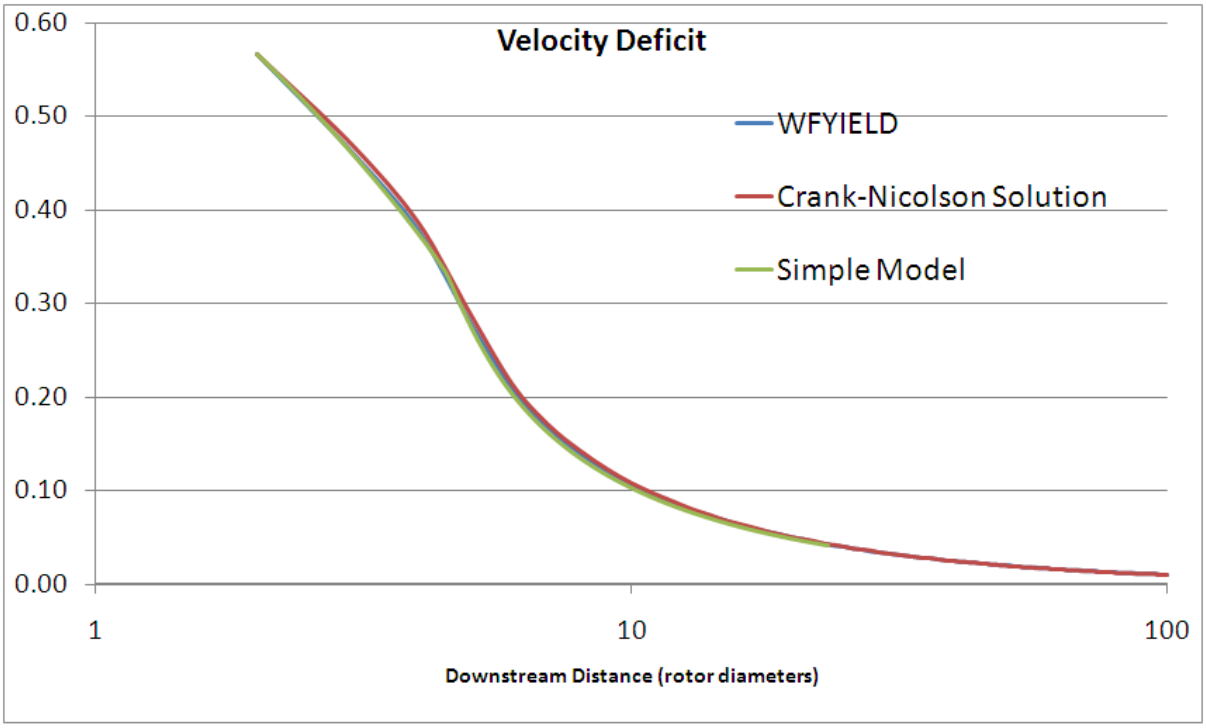

Figure 3 – Variation of Centreline Wake Deficit with Downstream Distance (Ct = 0.8, I = 15%).

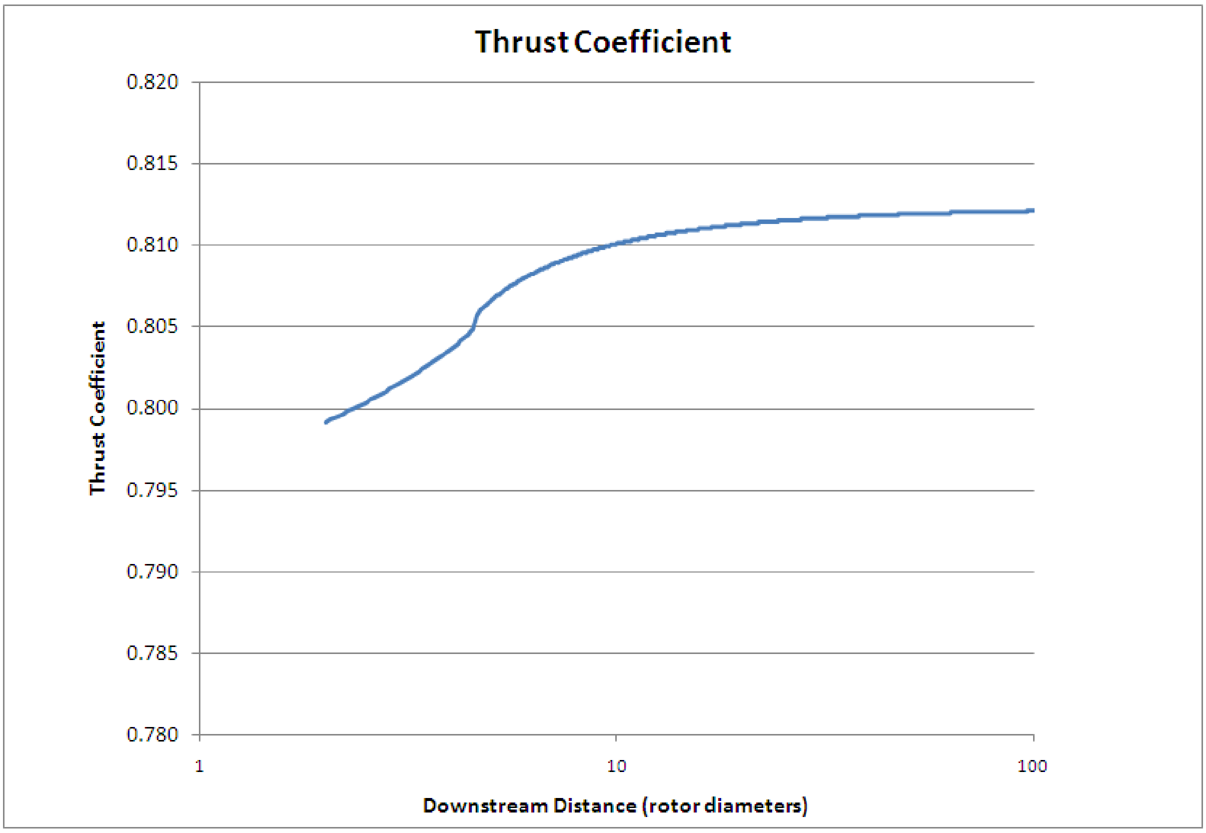

Figure 4 – Variation of Rotor Thrust Coefficient with Downstream Distance for the Crank-Nicolson method (Ct = 0.8, I = 10%).